Variance of transmitted power in multichannel dissipative ergodic structures invariant under time reversal

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We use random matrix theory (RMT) to study the first two moments of the wave power transmitted in time-reversal invariant systems having ergodic motion. Dissipation is modeled by a number of loss channels of variable coupling strength. To make a connection with ultrasonic experiments on ergodic elastodynamic billiards, the channels injecting and collecting the waves are assumed to be negligibly coupled to the medium and to contribute essentially no dissipation. Within the RMT model we calculate the quantities of interest exactly, employing the supersymmetry technique. This approach is found to be more accurate than another method based on simplifying naive assumptions for the statistics of the eigenfrequencies and the eigenfunctions. The results of the supersymmetric method are confirmed by Monte Carlo numerical simulation and are used to reveal a possible source of the disagreement between the predictions of the naive theory and ultrasonic measurements.

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I. INTRODUCTION

The statistics of waves in complex disordered and ray-chaotic structures have been well modeled in recent years by random matrix theory (RMT). The eigenstatistics of such structures are ergodically equivalent to those of certain classes of random matrices. This has been established by an enormous literature, both experimental and theoretical, and applies to the complex dynamics of compound nuclei [1], and also to the somewhat simpler case of closed nondissipative wave billiards with chaotic ray trajectories [2]. More recently attention has turned to the case of structures with open loss channels and/or internal dissipation [3–8]. This evolution of focus has been dictated by the physical systems available, for which it is difficult or impossible to eliminate absorption, and/or minimize the dissipative effect of the channels used to launch or detect the waves.

Of the many statistics that could be considered for such structures, perhaps the simplest experimentally accessible one is the relative variance of the power transmission. This quantity is related to cross section fluctuations in nuclear reactions, it is accessible in microwave experiments, and is of long standing interest in acoustics [8,9]. In Ref. [9] relative variances measured in a dissipative three-dimensional ultrasonic billiard were compared to the predictions of a simple theory which assumes that the eigenstatistics are identical to those of the nondissipative case. Such an assumption is strictly true only for very special cases of dissipation. The theory was found to consistently overestimate the relative variance of the mean square transmitted signal. Our chief interest here is to develop a more rigorous theory for that variance and to compare it with the predictions of the naive theory.

As an adequate theoretical model of such a structure we will use an effective random matrix theory description, with a random matrix H replacing the wave equation's linear differential operator (see Ref. [4] and references therein for a more detailed discussion). Then the complex amplitude of the transmitted wave is simply proportional to the ijth matrix element of the resolvent associated with the wave equation $G(E)\equiv[EI+i\varepsilon I-H+i\Gamma]^{-1}$. Here, I is the identity matrix, the matrix Γ describes coupling to internal local-in-space dissipative channels, the parameter $\varepsilon>0$ describes uniform losses, and E is the spectral variable. The same model describes microwave billiards, ultrasonic billiards, and nuclear reactions. The real symmetric positive semidefinite loss operator Γ can be written in terms of the states of the channels ($\Gamma=W^{\dagger}W$ in definitions of Ref. [10]) or in terms of absorption mechanisms. It is generally taken to be only weakly dependent on E. Thus both open and closed dissipative systems are described by the same model. When losses are negligible the experimental systems are usually invariant under time reversal. The appropriate choice for the corresponding random matrix H should, therefore, be a real symmetric matrix taken from the Gaussian orthogonal ensemble (GOE).

Our quantity of prime interest is $T=|G_{ij}(E)|^2, i\neq j$, i.e., the product of retarded and advanced Green's functions (propagators): $G^R_{ji}(E)\equiv[EI+i\varepsilon I-H+i\Gamma]^{-1}_{ji}$ and $G^A_{ij}=(G^R_{ji}(E))^*$, respectively. Except for slowly varying factors of receiver gain and source strength, the quantity T represents the ultrasonic power of Ref. [9].

For general nonperturbative statistical studies the only generally applicable tool known at present is reduction to the Efetov's zero-dimensional supersymmetric nonlinear σ model [11]. In this way the problem of calculating RMT ensemble averages reduces to performing a definite finite-dimensional integral over a space of supermatrices. The zero-dimensional σ model can also be derived from the assumptions of RMT [10], and is applicable to a variety of quantum-scattering problems formulated in terms of random Hamiltonians [8–12]. Ideally, once the quantity of interest is expressed in terms of the products of resolvents of the effective Hamiltonian $H-i\Gamma$, its mean, its variance, and sometimes its distribution function can be obtained.

Technical details of the supersymmetric reduction procedure depend essentially on the basic symmetries of the underlying ensemble. It is well known that working with the orthogonal ensemble leads to calculations, which are more technically involved than those of similar calculations for systems with broken time-reversal invariance (TRI). In the latter case, the corresponding ensemble is the Gaussian unitary (GUE), and one can go as far as calculating the full distribution function of transmitted wave power in ergodic systems [4]. Unfortunately, the existing experimental results on power transmission are only available for systems with preserved TRI.

The main goal of the present work is to explore transmitted power statistics for ergodic TRI systems. We find that it is possible to derive explicit analytical expressions for the first two moments of this quantity. We wish in particular to explain the differences, seen in Ref. [9], between the predictions of the oversimplified (''naive'' perturbative) theory and experimental measurements.

In Sec. II we use the supersymmetry method to derive expressions for mean and variance of transmitted power. In Sec. III, we confirm the results by numerical Monte Carlo calculation, and compare them with the results of the perturbative method of Refs. [4,9]. In Sec. IV we investigate a hypothesis to explain the longstanding discrepancy between lab measurements in ergodic acoustic systems and naive RMT predictions. A summary is given in Sec. V.

II. SUPERSYMMETRIC CALCULATION FOR MEAN AND MEAN SQUARED POWER

A. The system

In an ergodic system characterized by a random $N\times N$ Hamiltonian H and a dissipation matrix Γ, a matrix element of $G(E)=[EI+i\varepsilon I-H+i\Gamma]^{-1}$ represents the response spectrum (with E being the spectral variable). Its squared absolute value $G_{ij}(E)G^{*}_{ij}(E)$ denotes the spectral power density.

The elements of the random matrix H are zero-centered Gaussian variables, and because we deal with power transmission inside time-reversal invariant systems, the matrix H is real and symmetric. The relevant random matrix ensemble is, therefore, the GOE. Because of the inherent orthogonal invariance the dissipation matrix may be chosen to be diagonal: $\Gamma=\mathrm{diag}\{\gamma,\gamma,\ldots,\gamma,0,\ldots,0\}$, as we always can express our matrices in Γ's natural basis. The number $M<N$ of nonzero entries can be interpreted either as a number of equivalent open channels in the scattering system or, as a number of equivalent localized ''dampers'' in a closed system with losses [9]. Note, that convergence generating parameter $\varepsilon>0$ can be interpreted as the coupling to infinite number of external channels or, as uniform dissipation.

We are interested in the statistics of the wave power T transmitted from a source at point j to a receiver at a different point i inside the system: $T=G_{ij}(E)G^{*}_{ij}(E)$, $1\leqslant i,j\leqslant N\, j\neq i$ (no summation over i and j) [9].

B. Basic definitions and identities

To obtain expressions for the first two moments of the transmitted power $\overline{T}$, $\overline{T^2}$ (the bar indicates the ensemble averaging) we adopt a procedure similar to that of Ref. [10]. We start with constructing a generating function Z for our quantities of interest by introducing four-component supervectors Φ_p,

$$\Phi_p^T=\{\chi_p^{*T},\chi_p^T,S_p(1)^T,S_p(2)^T\},\quad p=1,2,$$

where the components of N-dimensional vectors S are real commuting variables, the elements of the vectors χ are anticommuting variables (Grassmannian), and T stands for the vector transposition. Index p is used to distinguish between retarded $(p=1)$ and advanced $(p=2)$ Green's functions. The latter can be obtained from the generating functions: $Z_p(E,\mathfrak{J}_p)=\int[d\Phi_p]\exp\{(i/2)\mathfrak{L}_p(E,\Phi_p,\mathfrak{J}_p)\}$, where the ''actions'' $\mathfrak{L}_p$ are defined as $\mathfrak{L}_p(E,\Phi_p,\mathfrak{J}_p)=\Phi_p^{\dagger}(\mathfrak{D}_p+\mathfrak{J}_p)\Phi_p$ in terms of the block-diagonal 4×4 symmetric supermatrices [7–9,11],

$$\mathfrak{D}_p=(EI-H)\otimes L_p+i(\varepsilon I+\Gamma)\otimes\Lambda_p L_p,$$

$$L_1=\mathrm{diag}\{I_2,I_2\},\quad L_2=\mathrm{diag}\{I_2,-I_2\},$$

$$\Lambda_1=\mathrm{diag}\{I_2,I_2\},\quad \Lambda_2=\mathrm{diag}\{-I_2,-I_2\},$$

$$\mathfrak{J}_1=\mathrm{diag}\{0,0,J^{(1)},J^{(2)}\},\quad \mathfrak{J}_2=\mathrm{diag}\{0,0,J^{(3)},J^{(4)}\}.$$

Here $J^{(p)}$ are $N\times N$ symmetric source matrices, I_2 is 2×2 identity matrix, and the integration measure is just a product of independent differentials of commuting and anticommuting variables. The generating function for the power moments $T=\mathfrak{D}_{ij}^{-1}\mathfrak{D}_{ij}^{*-1}$ and $T^2=(\mathfrak{D}_{ij}^{-1}\mathfrak{D}_{ij}^{*-1})^2$ then can be shown to have the following representation:

$$Z(E,\mathfrak{J})=Z_1(E,\mathfrak{J}_1)Z_2(E,\mathfrak{J}_2)=\int[d\Phi]\exp\left\{\frac{i}{2}\mathfrak{L}(E,\Phi,\mathfrak{J})\right\},\tag{1}$$

in terms of 8×8 block-diagonal supermatrices $\mathfrak{D}=\mathrm{diag}\{\mathfrak{D}_1,\mathfrak{D}_2\}$, $L=\mathrm{diag}\{L_1,L_2\}$, $\Lambda=\mathrm{diag}\{\Lambda_1,\Lambda_2\}$, $\mathfrak{J}=\mathrm{diag}\times\{\mathfrak{J}_1,\mathfrak{J}_2\}$, and $\Phi=\{\Phi_1,\Phi_2\}$, $\mathfrak{L}(E,\Phi,\mathfrak{J})=\mathfrak{L}_1(E,\Phi_1,\mathfrak{J}_1)+\mathfrak{L}_2(E,\Phi_2,\mathfrak{J}_2)=\Phi^{\dagger}(\mathfrak{D}+\mathfrak{J})\Phi$.

The Gaussian integral over the supervectors in Eq. (1) can be also written as a superdeterminant

$$Z(E,\mathfrak{J})=\prod_{p=1,2}Z_p(E,\mathfrak{J}_p)=\prod_{p=1,2}\mathrm{Sdet}^{-1}(\mathfrak{D}_p+\mathfrak{J}_p).$$

Differentiating this expression with respect to elements of the symmetric source matrix $\mathfrak{J}$ one finds (cf. Refs. [10,12])

$$\frac{\partial^2 Z(E,\mathfrak{J}=0)}{\partial J_{ij}^{(1)}\partial J_{ij}^{(3)}}=T,\tag{2}$$

$$\frac{\partial^4 Z(E,\mathfrak{J}=0)}{\partial J_{ij}^{(1)}\partial J_{ij}^{(2)}\partial J_{ij}^{(3)}\partial J_{ij}^{(4)}}=T^2,\tag{3}$$

relating both T and T^2 to the Gaussian integrals over the supervector components. Using the shorthand notation $\langle\cdots\rangle_\Phi = \int[d\Phi](\cdots)\exp\{i\mathfrak{L}(E,\Phi)/2\}$, we can write

$$T = \langle F_1[\Phi]\rangle_\Phi, \tag{4}$$

$$T^2 = \langle F_2[\Phi]\rangle_\Phi, \tag{5}$$

where we introduced the following products of the commuting components of the supervectors:

$$F_1[\Phi] = S(1)_{1i}S(1)_{2i}S(1)_{1j}S(1)_{2j},$$

$$F_2[\Phi] = S(1)_{1i}S(1)_{2i}S(2)_{1i}S(2)_{2i}S(1)_{1j}S(1)_{2j} \times S(2)_{1j}S(2)_{2j}.$$

Now, we proceed with GOE averaging of the above expressions for the moments of the transmitted power. In what follows we use the overbar to denote the averaging over H with the weight $\exp\{-(N/4v^2)\mathrm{Tr}\,H^TH\}$, so that $\overline{H_{ij}H_{kl}} = (v^2/N)(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$, i.e., the ensemble averaging. It can be performed exactly with the help of the identity:

$$\overline{\exp\left\{\frac{i}{2}\Phi^\dagger(H\otimes L)\Phi\right\}} = \exp\left\{-\frac{v^2}{4N}\mathrm{Str}A^2\right\},$$

where we introduced a new 8×8 supermatrix: $A = L^{1/2}\Sigma_{i=1}^N \Phi_i\Phi_i^\dagger L^{1/2}$. The elements of the supermatrix A are labeled as follows:

$$A = \begin{pmatrix} A_{mn}^{11} & A_{mn}^{12} \\ A_{mn}^{21} & A_{mn}^{22} \end{pmatrix},$$

where $m,n = 1, \ldots, 4$. With the help of these notations we can express $\overline{T}$ and $\overline{T^2}$ in a unified form via the representations

$$\overline{\langle F_{1,2}[\Phi]\rangle_\Phi} = \int [d\Phi]F_{1,2}[\Phi]\exp\left\{\frac{i}{2}E\Phi^\dagger L\Phi - \frac{1}{2}\Phi^\dagger(\Gamma\otimes\Lambda)L\Phi - \frac{v^2}{4N}\mathrm{Str}A^2 - \frac{\varepsilon}{2}\mathrm{Str}A\Lambda\right\}$$

as both formulas differ only in the form of preexponent factors F.

C. Performing Φ integration

The next step of the supersymmetric calculation is the so-called Hubbard-Stratonovich decoupling [11,12],

$$\exp\left\{-\frac{v^2}{4N}\mathrm{Str}A^2 - \frac{\varepsilon}{2}\mathrm{Str}A\Lambda\right\} = \int [dR]\exp\left\{-\frac{N}{4}\mathrm{Str}R^2 + i\frac{\varepsilon}{2v}N\mathrm{Str}R\Lambda + i\frac{v}{2}\mathrm{Str}RA\right\},$$

$$\overline{\langle F_{1,2}[\Phi]\rangle_\Phi} = \int [dR]\exp\left\{-\frac{N}{4}\mathrm{Str}R^2 + i\frac{\varepsilon}{2v}N\mathrm{Str}R\Lambda\right\} \times \int [d\Phi]F_{1,2}[\Phi]\exp\left\{-\frac{i}{2}\Phi^\dagger L^{1/2}f^{-1}L^{1/2}\Phi\right\}, \tag{6}$$

where we defined $8N\times 8N$ supermatrix f,

$$f = [-EI\otimes I_8 - vI\otimes R - i(\Gamma\otimes\Lambda)]^{-1} = [(I_N\otimes I_8 - i\Gamma\otimes(\Lambda\mathfrak{G}^{-1}))\mathfrak{G}]^{-1},$$

with $\mathfrak{G} = -EI_8 - vR$. In Eq. (6) we can integrate out Φ variables, using Wick's theorem for supervectors, and bring the remaining integral into a form suitable for a saddle-point approximation in the limit $N\to\infty$ [11,12]. Then for $i,j > M$ we obtain

$$\int [d\Phi]F_{1,2}[\Phi]\exp\left\{-\frac{i}{2}\Phi^\dagger f^{-1}\Phi\right\} = F_{1,2}[\mathfrak{G}^{-1}](\mathrm{Sdet}f)^{1/2}. \tag{7}$$

Here we introduced the notations

$$F_1[\mathfrak{G}^{-1}] = \frac{1}{4}\{(\mathfrak{G}^{-1})_{33}^{12} + (\mathfrak{G}^{-1})_{33}^{21}\}^2, \tag{8}$$

and

$$F_2[\mathfrak{G}^{-1}] = \{(\mathfrak{G}_+^{-1})_{34}^{11}(\mathfrak{G}_+^{-1})_{34}^{22} + (\mathfrak{G}_+^{-1})_{33}^{12}(\mathfrak{G}_+^{-1})_{44}^{12} + (\mathfrak{G}_+^{-1})_{34}^{12}(\mathfrak{G}_+^{-1})_{43}^{12}\}, \tag{9}$$

where $\mathfrak{G}_+^{-1} = \{\mathfrak{G}^{-1} + (\mathfrak{G}^{-1})^T\}/2$. At this point we summarize the results for $\overline{T}$ and $\overline{T^2}$ separately,

$$\overline{T} = \int [dR]F_1[\mathfrak{G}^{-1}]\exp\{-N\mathcal{L}[R] + \delta\mathcal{L}\}, \tag{10}$$

$$\overline{T^2} = \int [dR]F_2[\mathfrak{G}^{-1}]\exp\{-N\mathcal{L}[R] + \delta\mathcal{L}\}, \tag{11}$$

where the exponential is given by

$$\mathcal{L}[R] = \frac{1}{4}\mathrm{Str}R^2 + \frac{1}{2}\mathrm{Str}\ln(-EI_8 - vR), \tag{12}$$

$$\delta\mathcal{L} = i\frac{\varepsilon}{2v}N\,\mathrm{Str}\,R\Lambda - \frac{M}{2}\mathrm{Str}\ln[I_8 - i\gamma\Lambda(-EI_8 - vR)^{-1}]. \tag{13}$$

The remaining step is to carry out integration in Eqs. (10) and (11) by the saddle-point method in the limit of large N. The stationarity condition for $\mathcal{L}[R]$ yields the saddle-point equation $R_s = v/(-EI_8 - vR_s)$. Its solution is given by a saddle-point manifold in a space of 8×8 supermatrices [11,10],

$$R_s = -\frac{E}{2v} I_8 + i\pi\nu v \mathfrak{T}^{-1}\Lambda\mathfrak{T} = -\frac{E}{2v} I_8 - \pi v \nu Q. \quad (14)$$

Here ν denotes the mean eigenvalue density given for the GOE by the Wigner semicircular law $\nu = \sqrt{4v^2 - E^2}/(2\pi v^2)$. After integrating out the massive Gaussian fluctuations around the saddle-point manifold in Eqs. (10) and (11), the first two moments of the transmitted power are expressed as integrals over the supermatrices $Q = \mathfrak{T}^{-1}\Lambda\mathfrak{T}$ [11,10],

$$\frac{\overline{T}}{(\pi\nu)^2} = \langle F_1[Q]\rangle_Q$$
$$= \int [dQ] F_1[Q] \mathrm{Sdet}^{-M/2}\left[I_8 + i\frac{E}{2v^2}\gamma\Lambda + i\pi\nu\gamma Q\Lambda\right] \times \exp\left\{ -\frac{i}{2}\varepsilon\pi\nu N \,\mathrm{Str}\, Q\Lambda\right\}, \quad (15)$$

$$\frac{\overline{T^2}}{(\pi\nu)^4} = \langle F_2[Q]\rangle_Q . \quad (16)$$

This step completes derivation of the zero-dimensional nonlinear σ model.

D. Performing Q integration

To evaluate the superintegrals in Eqs. (15) and (16), we need to calculate $F_1[Q]$ and $F_2[Q]$ first. At this point we employ the Verbaarschot-Weidenmueller-Zirnbauer parametrization [10] for the matrix Q. Both $F_1[Q]$ and $F_2[Q]$ are the functions of matrix elements of Q, obtained by the formal substitution of Q for G^{-1} in Eqs. (8) and (9), as follows from Eq. (14). Matrix elements of Q are, in turn, the functions of eight commuting and eight anticommuting variables. Although we are interested in the highest order terms in anticommuting variables [13], the calculation of $F_1[Q]$ and $F_2[Q]$ is too cumbersome to be done by hand. The calculation can be managed most efficiently by employing the symbolic manipulation package epicGRASS [14]. The outputs of the epicGRASS (the highest order terms in anticommuting variables) need to be further integrated over all the anticommuting variables, and finally over all the commuting variables except "eigenvalues" [10]. After those steps we change to the λ variables of Ref. [11], and arrive at the representation for T and T^2 in terms of a threefold integral. The details of this procedure are outlined in Appendix A. Here we only give the final expression

$$\frac{\overline{T}}{(\pi\nu)^2} = \int_1^\infty d\lambda_1 \int_1^\infty d\lambda_2 \int_{-1}^1 d\lambda F_1(\lambda,\lambda_1,\lambda_2) \times \exp\{-\epsilon(\lambda_1\lambda_2 - \lambda)\}\mu(\lambda,\lambda_1,\lambda_2)\Pi(\lambda,\lambda_1,\lambda_2), \quad (17)$$

$$\frac{\overline{T^2}}{(\pi\nu)^4} = \int_1^\infty d\lambda_1 \int_1^\infty d\lambda_2 \int_{-1}^1 d\lambda F_2(\lambda,\lambda_1,\lambda_2) \times \exp\{-\epsilon(\lambda_1\lambda_2 - \lambda)\}\Pi(\lambda,\lambda_1,\lambda_2)\mu(\lambda,\lambda_1,\lambda_2), \quad (18)$$

where $\epsilon = 2\pi\nu N\varepsilon$, and

$$\mu(\lambda,\lambda_1,\lambda_2) = \frac{1-\lambda^2}{(\lambda_1^2+\lambda_2^2+\lambda^2-2\lambda\lambda_1\lambda_2-1)^2},$$

$$F_1(\lambda,\lambda_1,\lambda_2) = 1-\lambda^2+(\lambda_1^2-1)\lambda_2^2+(\lambda_2^2-1)\lambda_1^2,$$

$$F_2(\lambda,\lambda_1,\lambda_2) = 2(1-\lambda_1^2-\lambda_2^2-2\lambda\lambda_1\lambda_2+3\lambda_1^2\lambda_2^2)^2.$$

The remaining factor $\Pi(\lambda,\lambda_1,\lambda_2)$ contains all the information about the dissipation channels and comes from a calculation of the relevant superdeterminant (cf. Ref. [15] for the GUE case)

$$\mathrm{Sdet}^{-M/2}\left[I_8 + i\frac{E}{2v^2}\gamma\Lambda + i\pi\nu\gamma Q\Lambda\right] = \left(\frac{v^2+\gamma^2+2\pi\nu\gamma\lambda}{\sqrt{(v^2+\gamma^2)^2+4\pi\nu\gamma v^2(v^2+\gamma^2)\lambda_1\lambda_2+(2\pi\nu\gamma v^2)^2(\lambda_1^2+\lambda_2^2-1)}}\right)^M$$
$$= \frac{(g+\lambda)^M}{(\sqrt{g^2+2g\lambda_1\lambda_2+\lambda_1^2+\lambda_2^2-1})^M}$$
$$= \Pi(\lambda,\lambda_1,\lambda_2), \quad (19)$$

where $g = (v^2+\gamma^2)/(2\pi\nu\gamma v^2)$, and we have also used

$$\mathrm{Str}\, Q\Lambda = -4i(\lambda_1\lambda_2 - \lambda). \quad (20)$$

To generalize Eqs. (17) and (18) to the case of nonequipotent dampers we just need to replace $\Pi(\lambda,\lambda_1,\lambda_2)$ with

$$\Pi(g_i,\lambda,\lambda_1,\lambda_2) = \prod_i \frac{(g_i+\lambda)}{(\sqrt{g_i^2+2g_i\lambda_1\lambda_2+\lambda_1^2+\lambda_2^2-1})},$$

see, for example, Refs. [10,16]. It can be verified that Eq.

(17) yields the same result for $\overline{T}$ as follows from adopting the final formula of Ref. [10].

E. Special case of uniform damping: Comparison with naive calculation

Next, we compare results of the present (supersymmetric) calculation with the results of Ref. [9] for the case of uniform damping $M=0,\epsilon\neq 0$. In that special case the naive calculation of Ref. [9] should be exact. In order to obtain $\overline{T}$ we need to evaluate the integral

$$I(x)=\int_1^\infty d\lambda_1\int_1^\infty d\lambda_2\int_{-1}^1 d\lambda \exp\{ix(\lambda_1\lambda_2-\lambda)\}\times F_1(\lambda,\lambda_1,\lambda_2)\mu(\lambda,\lambda_1,\lambda_2), \tag{21}$$

where we have denoted $x=i\epsilon$. The Fourier transformation with respect to the x variable,

$$\tilde{I}(t)=\int_{-\infty}^{\infty} I(x)\exp\{-ixt\}dx,$$

has a meaning of averaged response power in the time domain for a system without dissipation. It can be written in a more convenient form

$$\tilde{I}(t)=2\pi\int_1^\infty d\lambda_1\int_1^\infty d\lambda_2\int_{-1}^1 d\lambda\,\delta(\lambda-\lambda_1\lambda_2+t)\times F_1(\lambda,\lambda_1,\lambda_2)\mu(\lambda,\lambda_1,\lambda_2). \tag{22}$$

After performing λ integration, we make the change of variables: $u=\lambda_1\lambda_2, z=\lambda_1^2$ suggested in Ref. [11], and after a lengthy but straightforward procedure arrive at a very simple expression

$$\tilde{I}(t)=4\pi\theta(t).$$

which can be immediately Fourier inverted, yielding

$$I(x)=\frac{-2i}{x}.$$

This is equivalent to the first moment of the transmitted power given by

$$\frac{\overline{T}}{(\pi\nu)^2}=\frac{2}{\epsilon}, \tag{23}$$

and indeed coincides with the value predicted by the naive calculation of Ref. [9].

The same steps can be repeated when calculating the second moment $\overline{T^2}$. One starts with Fourier-transforming the right-hand side of Eq. (18), then changes to the variables u and z, carries out the remaining double integration explicitly and finally applies the Fourier inversion. Intermediate calculations are too long to be reproduced in the paper, but the final result reads

$$\begin{aligned}\frac{\overline{T^2}}{(\pi\nu)^4}=&\frac{1}{\epsilon^4}(5+28\epsilon+7\epsilon^2)-\frac{e^{-2\epsilon}}{\epsilon^4}(5+2\epsilon+\epsilon^2)\\&+\frac{e^{-\epsilon}}{\epsilon^4}E_1(\epsilon)(10+10\epsilon+3\epsilon^2+\epsilon^3)\\&+\frac{e^{\epsilon}}{\epsilon^4}E_1(\epsilon)(-10+10\epsilon-3\epsilon^2+\epsilon^3),\end{aligned} \tag{24}$$

where

$$E_1(z)=\int_z^\infty \frac{e^{-s}}{s}ds.$$

This matches perfectly with the corresponding result of Ref. [9].

III. NUMERICAL RESULTS FOR THE MOMENTS OF THE TRANSMITTED POWER

The predictions Eqs. (17) and (18) of the supersymmetric calculations can be compared with Monte Carlo evaluations of the first two moments of T. Towards this goal we numerically generated an ensemble of $N\times N$ real symmetric matrices H typically choosing 1500 ensemble realizations and taking $N=1000$. The procedure is almost identical to that described in Ref. [4]. The entries in H are constructed using a Gaussian random number generator such that $\overline{H_{ij}H_{kl}}=(1/N)(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})$. To simulate the case of the uniform damping we use $\Gamma=\varepsilon I$. To simulate the case of a finite number of decay channels we take the diagonal $\Gamma=\mathrm{diag}\{\gamma,\gamma,\ldots,\gamma,0,\ldots,0\}$ with $M<N$ identical positive entries. Then, for every ensemble realization we generate the off-diagonal elements of the resolvent matrix according to $G_{ij}(E)=[EI+i\Gamma-H]^{-1}$, modeling in this way the response at a site i due to excitation at the site j, with E standing for the spectral parameter, and both i and j chosen to be larger than M to avoid direct coupling to the damping channels.

Let us first consider the case of the uniform damping: $\Gamma=\varepsilon I$. For a fixed matrix size N and fixed value of the spectral variable E we explore a range of ε. For $E=0$ the modal density $\partial N/\partial E$ is given by $\nu=1/\pi$. Mean level width $\overline{\gamma}=2\pi\nu\langle\Im E_r\rangle$ in this case is identical to $\epsilon=2\pi\nu N\varepsilon$. In Fig. 1 we compare both moments of power T as given by Eqs. (23) and (24) with the results of Monte Carlo simulations for several values of ϵ. It is evident that numerical results correspond well with the theoretical curves.

To repeat the same procedure for finite number of local dampers M we evaluated the three-dimensional integrals in Eqs. (17) and (18) numerically for a broad range of the scaled mean level width $\overline{\gamma}$ [4]. The difficulties of the numerical integration arising due to the singularity of $\mu(\lambda,\lambda_1,\lambda_2)$, are easy to overcome by employing the change of variables suggested in Ref. [17]. The results are presented in Fig. 2 and also show a good agreement with the theory.

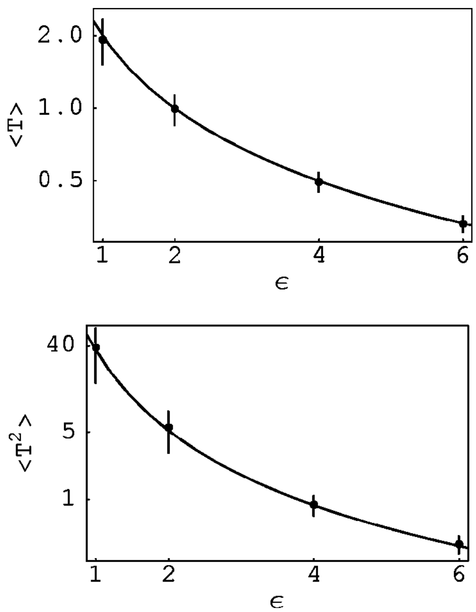

FIG. 1. $\langle T\rangle$ and $\langle T^2\rangle$ are plotted on log scale as functions of the parameter ϵ for the case of uniform damping. The solid lines represent theoretical predictions [Eqs. (17) and (18)]. For each numerically obtained $\langle T\rangle$ and $\langle T^2\rangle$(represented by dots) 1500 samples of $|G_{ij}(E)|^2$, $i\neq j$, were computed. 5σ error bars were computed based on the observed variances of T and T^2.

IV. RELATIVE VARIANCE

Dissipation within the framework of the present approach is parametrized in terms of quantities g, M, and ϵ. At the same time those quantities are not readily accessible experimentally, and in any case were not measured in the work reported in Ref. [9]. For this reason any direct comparison with those measurements is not feasible. Nevertheless, by choosing plausible values for the relevant parameters we can investigate the sign and magnitude of the discrepancies arising between the predictions for the relative variance of the transmitted power calculated in the two theories under discussion. As a result of such comparison we found that the discrepancy between the naive analysis and the present (supersymmetric) calculation is similar to one reported previously in Ref. [9] between the naive theory and actual measurements.

The comparison is carried out by first considering the mean spectral energy density (mean square response) in the time domain: $E(t)=|G_{ij}(t)|^2$, where $G_{ij}(t)$ is the (band-limited) time-domain Green's function. Similar quantities were studied previously in the context of the delay time distributions in chaotic cavities [18]. Their statistics can be obtained from the inverse Fourier transform of the two-point correlation function $T(\Omega)=\overline{G_{ij}(E)G^*_{ij}(E+\Omega)}$ [10,17] with respect to Ω. The expression for $T(\Omega)$ can be obtained by replacing $2i\varepsilon$ with Ω in the derivation of Eq. (17) (see also Ref. [17]). Thus,

$$E(\tau)\sim\int_1^\infty\int_1^\infty d\lambda_1 d\lambda_2 \Pi(\tau,\lambda_1,\lambda_2) f(\tau,\lambda_1,\lambda_2) \times\frac{\theta(\lambda_1\lambda_2-\tau+1)\,\theta(\tau-\lambda_1\lambda_2+1)(1-(\tau+\lambda_1\lambda_2)^2)}{(\lambda_1^2+\lambda_2^2+(\tau+\lambda_1\lambda_2)^2-2\lambda_1\lambda_2(\tau+\lambda_1\lambda_2)-1)^2}, \tag{25}$$

where $\tau=t/(2\pi\nu N)$ is a dimensionless time, $f(\tau,\lambda_1,\lambda_2)=(\lambda_1^2-1)\lambda_2^2+(\lambda_2^2-1)\lambda_1^2+1-(\tau+\lambda_1\lambda_2)^2$, and $\Pi(\tau,\lambda_1,\lambda_2)=(g+2\tau+\lambda_1\lambda_2)^M(g^2+2g\lambda_1\lambda_2+\lambda_1^2+\lambda_2^2-1)^{-M/2}$. The generalization to the case of nonequipotent channels is straightforward, $\Pi(\tau,\lambda_1,\lambda_2)=\Pi_i^M(g_i+2\tau+\lambda_1\lambda_2)(g_i^2+2g_i\lambda_1\lambda_2+\lambda_1^2+\lambda_2^2-1)^{-1/2}$. The naive method yields a simpler expression for spectral energy density [9,18,19],

$$E(\tau)_{naive}=E_0\left(1+\frac{2\sigma\tau}{M}\right)^{-M/2}, \tag{26}$$

where the initial logarithmic decay rate σ is proportional to the mean resonance width, given by a Porter-Thomas distribution.

In Ref. [9] $E(\tau)$ was measured experimentally, and fitted into the naive result (26) to extract values for M and σ. The two parameters were further used to predict the relative variance of T (relative variance$=\langle T^2\rangle/\langle T\rangle^2-1$). Having the exact result [Eq. (25)] we can now attempt to explain the 20–30% overprediction of relative variance reported in Ref. [9]. Clearly, by specifying certain values for M, g, and ϵ the wave scattering in an ergodic sample can be completely described, since both spectral energy density and relative variance are fixed uniquely. Further assuming that $E(\tau)$ as given by Eq. (25) is the "measured" energy density of our system, we can repeat the procedure of Ref. [9]. Namely, we fit it to $E(\tau)_{naive}$ in order to calculate relative variance according to the two-parameter naive formula used in Ref. [9] for comparison with actual measurements. Such a fit allows us to extract values for M_{naive}, σ_{naive}, and E_0 that may or may not correspond to the exact values. The true value of relative variance as determined from Eqs. (17) and (18) may then be compared to the corresponding naive prediction.

By a numerical three-parameter fit over the same dynamic range (of a factor of e^{10}) as in Ref. [9], we obtained values for E_0, σ_{naive}, and M_{naive}. In spite of the naivete of the model the fits were generally quite good, as observed in Ref. [9], and we can substitute the obtained values into the formula for the relative variance from Ref. [9],

$$\frac{\langle T^2\rangle}{\langle T\rangle^2}-1=1+\frac{9}{\sigma}\frac{M(M-2)}{(M-4)(M-6)}-4\left\{i_1+\sigma^2\frac{(M-2)^2}{M^2}i_2\right\}, \tag{27}$$

where

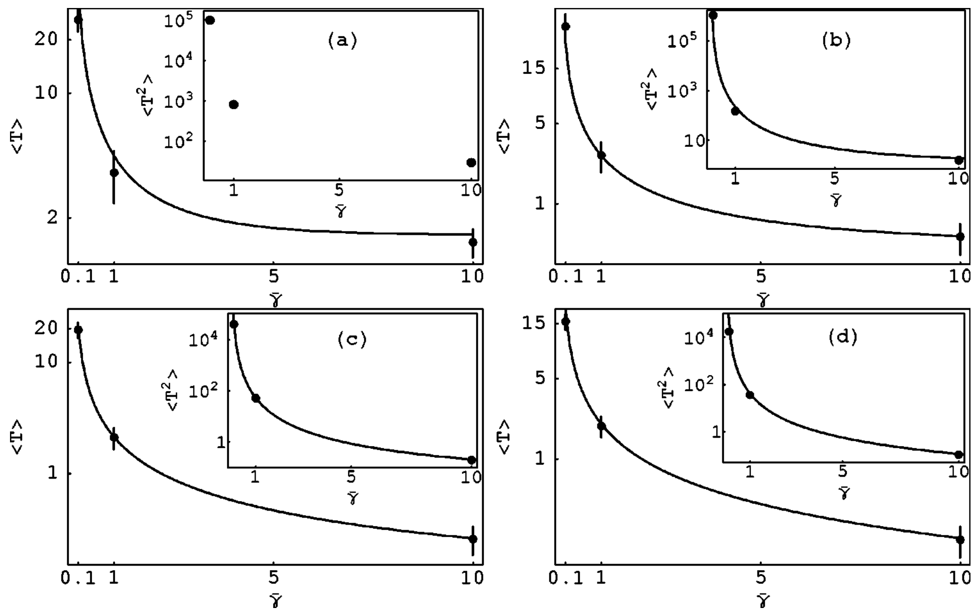

FIG. 2. Mean power (main figure) and mean square power (inset) for (a) $M=4$, (b) $M=10$, (c) $M=40$, (d) $M=400$. Solid lines represent theoretical predictions [Eqs. (17) and (18)]. For each numerically obtained $\langle T\rangle$ and $\langle T^2\rangle$ (dots) 1500 samples of $|G_{ij}(E)|^2$ were computed. We imposed the restrictions $i\neq j$, and $i>M, j>M$ for the nonuniform damping case, to avoid "recording" the response from damped sites or from the "source" site j, and to correspond to the assumptions in the theoretical analysis. For the numerically obtained mean power, 20σ, 10σ, and 5σ error bars were computed for the scaled mean level width $\overline{\gamma}=0.1,1.0,10.0$, respectively. They were based on the observed variances of T and T^2. Error bars for the power variance are not shown, because they are smaller than the size of the dots. For $M=4$ the theoretical prediction for the variance does not exist.

$$i_1=\frac{M}{2\sigma}\exp\left\{\frac{M}{\sigma}\right\}E_{M-2}\left(\frac{M}{\sigma}\right),$$

$$i_2=\frac{M^3}{8\sigma^3}\exp\left\{\frac{M}{\sigma}\right\}\left\{E_M\left(\frac{M}{\sigma}\right)-2E_{M-1}\left(\frac{M}{\sigma}\right)+E_{M-2}\left(\frac{M}{\sigma}\right)\right\},$$

$$E_k(z)=\int_1^\infty \frac{e^{-zs}}{s^k}ds.$$

The basic steps leading to Eq. (27) are explained in Appendix B.

The results for several values of parameters are summarized in Tables I and II. It appears that in the absence of overall damping (Table I) the actual value of relative variance is very close to its naive estimate. However, when we consider the case of a small number of strong dampers in a system with a uniform background $\epsilon\neq 0$ (Table II), the difference becomes similar to the discrepancy found in Ref. [9]. A more definitive comparison of Eqs. (17) and (18) with measurements awaits an experiment in which the values of ϵ and the g_i can be ascertained independently.

TABLE I. Relative variance in absence of overall damping.

M	g	σ	Naive	Exact
10	20.017	0.497	59.881	59.492
20	20.017	0.989	14.419	14.397
20	10.033	1.930	7.582	7.574
14	2.918	4.066	4.990	4.776

V. CONCLUSIONS

In the present paper the special cases of two and four point correlation functions of the transmitted power spectrum have been calculated both analytically and numerically for ergodic dissipative structures. In the context of the wave scattering the former corresponds to the mean and the latter to the mean square of the transmitted wave power T.

The ergodicity assumption is implicit by virtue of our replacement of the actual differential operator describing wave motion by a large random symmetric matrix. Dissipation is taken to act both locally in space ("localized dampers" or dissipative channels) and uniformly within the sample.

In accord with earlier results [4], the presence of nonuniform, or localized, sources of dissipation requires the use of an elaborate nonperturbative technique—the so-called zero-

TABLE II. Relative variance in presence of overall damping.

M	g	ϵ	σ	Naive	Exact
1	1.0	1.0	1.302	7.711	6.801
1	2.0	1.0	1.205	7.908	6.809
1	5.0	1.0	1.126	8.195	6.784
4	10.0	1.0	1.330	7.194	6.134
6	9.0	0.5	1.008	10.557	8.611

dimensional supersymmetric nonlinear σ model—to obtain the moments of the transmitted power. It is found that the naive approach fails to correctly describe mean square power; the failure is ascribable at least in part to the assumption of real Gaussian eigenmodes inherent in that approach. The supersymmetry technique allows one to bypass the difficulty of identifying eigenmode statistics, and to arrive at expressions which are in agreement with Monte Carlo simulations, and appear to be in better agreement with experimentally measured values of variance.

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APPENDIX A: EVALUATION OF THE SUPERINTEGRAL

In this appendix we elaborate on steps leading to the main results of Sec. II, expressed by Eqs. (17) and (18). We start by evaluating $F_1[Q]$ and $F_2[Q]$ with EPICGRASS. The program extracted terms of lowest and highest order in anticommuting variables, which are, generally, the only terms needed. The lowest order term was found to be unimportant since the resulting integrands are not singular at the boundary [11,13]. Then, we simplified the output of EPICGRASS with MATHEMATICA and reduced the superintegral to a multiple integral over commuting and anticommuting variables [10].

The elements of matrix Q are introduced into the EPICGRASS in terms of the parametrization of Ref. [10]. Eight commuting variables are: the eigenvalues μ_1, μ_2, μ, the parameters of SU(2) group m, r, s, and two angles φ_1 and φ_2. The integration region in Eqs. (15) and (16) corresponds to $-\infty < \mu_1, \mu_2, m, r, s < \infty$, $0 < \mu < 1$, $0 < \varphi_1, \varphi_2 < 2\pi$ [20]. Then, after EPICGRASS extracts the highest order term in anticommuting variables, we have, for example, for $F_1[Q]$,

$$\begin{aligned}F_1[Q] = &-32z^2 - 32z_1z_2\cos\varphi_1\cos\varphi_2\sin\varphi_1\sin\varphi_2\\ &-z_1^2(36\cos\varphi_1^2\cos\varphi_2^2 + 12\cos\varphi_1^2\sin\varphi_2^2\\ &+12\cos\varphi_2^2\sin\varphi_1^2 + 4\sin\varphi_1^2\sin\varphi_2^2)\\ &-z_2^2(36\sin\varphi_1^2\sin\varphi_2^2 + 12\cos\varphi_1^2\sin\varphi_2^2\\ &+12\cos\varphi_2^2\sin\varphi_1^2 + 4\cos\varphi_1^2\cos\varphi_2^2),\end{aligned}$$

where $z_{1,2} = \mu_{1,2}\sqrt{1+\mu_{1,2}^2}$ and $z = i\mu\sqrt{1-\mu^2}$ [21,22].

Eight anticommuting variables are readily integrated out according to the convention $\int d\chi\chi = 1/(2\pi)^{1/2}$. Note that this convention is different from the one we took in the beginning of Sec. II in the derivation of generating function. However, this discrepancy has no influence on the remaining process, as long as we use the integration measure of Ref. [10]. Finally, integrating over the angles as well as over the parameters of SU(2) we arrive at

$$\tilde{F}_1[Q] = -16(z_1^2 + z_2^2 - 2z^2), \tag{A1}$$

where we indicated the integration [which does not affect other factors in the integrand in Eq. (15)] by tilde.

Upon the substitution of eigenvalues into Eq. (A1) we can compare our expression for $\overline{T}$ with the final formula of Ref. [10]. We switch to the combinations

$$\lambda_{1,2}^V = \mu_{1,2.}^2, \quad \lambda^V = \mu^2,$$

which are the final variables appearing in the resulting expression of Ref. [10]. Two results match perfectly, and we can proceed with the corresponding calculation of the second moment of the transmitted power. Before doing that we again change variables, this time—to the eigenvalues of Efetov's parametrization, according to

$$\lambda_{1,2}^V = \lambda_1\lambda_2 \pm \sqrt{(\lambda_1^2-1)(\lambda_2^2-1)}.$$

The domain of the integration has to be modified as well: $1 < \lambda_1, \lambda_2 < \infty, -1 < \lambda < 1$. The Efetov's eigenvalues are somewhat more convenient for the calculations done at the end of Sec. II, where we compared the exact and naive results for the first two moments in uniform damping case.

The analogous procedure for $\overline{T^2}$ yields

$$\begin{aligned}\tilde{F}_2[Q] = 4(&4x^2 - 4xx_1 - 4xx_2 + x_1^2 + x_2^2\\ &+2x_1x_2 + 8z_1^2 + 8z_2^2 - 16z^2)^2,\end{aligned} \tag{A2}$$

where $x_{1,2} = 1 + 2\mu_{1,2}^2$, $x = 1 - 2\mu^2$, and after passing to Efetov's variables in Eqs. (A1) and (A2) we obtain the final results of Sec. II—Eqs. (17) and (18).

APPENDIX B: DETAILS OF THE NAIVE CALCULATION

The method of Ref. [9] leading to Eq. (27) is presented here for completeness. Mean square power is given by a modal expansion (mean power requires a similar though simpler calculation),

$$\begin{aligned}T^2 = \sum_{r,m,l,k} &\frac{u_i^r u_j^{r*}}{E - E_r - i\zeta_r}\frac{u_i^{m*} u_j^m}{E - E_m + i\zeta_m}\\ &\times\frac{u_i^{l*} u_j^l}{E - E_l - i\zeta_l}\frac{u_i^{k*} u_j^k}{E - E_k + i\zeta_k}.\end{aligned} \tag{B1}$$

We make assumptions about a lack of correlations amongst modal amplitudes and frequencies and widths, and conclude

$$\langle T^2\rangle=\sum_r \frac{\langle u^4\rangle^2}{(E-E_r-i\zeta_r)^2(E-E_r+i\zeta_r)^2}+\sum_{r\neq l}\frac{\langle u^2\rangle^4}{(E-E_r-i\zeta_r)^2(E-E_r+i\zeta_l)^2}$$
$$+\sum_{r\neq l}\frac{\langle u^2\rangle^4}{(E-E_r-i\zeta_r)(E-E_r+i\zeta_r)(E-E_r-i\zeta_r)(E-E_r+i\zeta_r)}. \tag{B2}$$

The summations over the modes are replaced with integrations ($\Sigma_r \rightarrow N\nu\int dE_r$), and eigenfrequency correlations are taken to be those of GOE, by introducing the Dyson two level function Y_2 [12]. We have

$$\langle T^2\rangle=\frac{N\nu\langle u^4\rangle^2}{(\pi\nu)^4}\int_{-\infty}^{\infty}\frac{dx}{(x-i\zeta_r)^2(x+i\zeta_r)^2}+\frac{2(N\nu)^2\langle u^2\rangle^4}{(\pi\nu)^4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{[1-Y_2(\pi N\nu z)]dx\,dz}{(x^2+\zeta_r)[(x-z)^2+\zeta_l^2]}$$
$$+\frac{(N\nu)^2\langle u^2\rangle^4}{(\pi\nu)^4}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{[1-Y_2(\pi N\nu z)]dx\,dz}{(x-i\zeta_r)^2(x-z+i\zeta_l)^2}, \tag{B3}$$

where $x=E-E_r$, $z=E_r-E_l$. The remaining steps of the naive ensemble averaging procedure include integration over a distribution of widths, given by [12]

$$p\left(\frac{\zeta_r}{\overline{\Gamma}}\right)=\frac{(M/2)^{M/2}}{\Gamma(M/2)}\left(\frac{\zeta_r}{\overline{\Gamma}}\right)^{M/2-1}\exp\left\{-M\frac{\zeta_r}{2\overline{\Gamma}}\right\}, \tag{B4}$$

where $\overline{\Gamma}$ is average resonance width and $\Gamma(s)$ is the Gamma function,

$$\langle T^2\rangle=\overline{N\nu\langle u^4\rangle^2\frac{\pi}{\zeta_r^3}}+\overline{2(N\nu)^2\langle u^2\rangle^4 I},$$

$$I=\frac{\zeta_r+\zeta_l}{2\zeta_r\zeta_l}\int_{-\infty}^{\infty}\frac{(1-Y_2(\pi N\nu z))dz}{z^2+(\zeta_r+\zeta_l)^2}.$$

The average over ζ_r and ζ_l is indicated by overbar. After substituting $\langle u^2\rangle=1/N$, $\langle u^4\rangle/\langle u^2\rangle^2=3$ (as for Gaussian random numbers) and integrating with respect to eigenwidths, z integration becomes straightforward and results in Eq. (27).

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